TWO-STRUCTURE FRAMEWORK FOR HAMILTONIAN DYNAMICAL SYSTEMS [*]

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Abstract

The Lie product and the order relation are viewed as defining structures for Hamiltonian dynamical systems. Their admissible combinations are singled out by the requirement that the group of the Lie automorphisms be contained in the group of the order automorphisms (Lie algebras with invariant cones). Taking advantage of the reciprocal independence of the relevant structures, the inclusion relation between the two automorphism groups can be reversed; a procedure which leads to an entirely new formal language (ordered linear spaces with invariant Lie products). Presumably it offers an alternative description for quantum systems, radically different from the conventional algebraic models.

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I. OBJECTIVES

We aim at expanding the scope of the Hamiltonian formalism as much as possible by appropriate mathematical definitions for basic concepts like variables, states etc. The project is motivated partly by the situation in the quantum theory where increasing pressure is felt to go beyond the limits of the non-commutative associative algebras. One expects that the non-commutative associative multiplication could be replaced by weaker mathematical structures without affecting too much the physical content of the theory. We propose a pair of structures that promise to get the job done.

II. LIE ALGEBRAS WITH INVARIANT CONES

Let us name at once the structures we are going to use: Lie algebras and order relations. We view the space of the variables of the Hamiltonian systems as ordered Lie algebras, or (which is the same thing) as Lie algebras with a fixed (positive) cone.

The Lie algebra structure is inherent in all the known classes of Hamiltonian systems. On the other hand, given any (abstract) Lie algebra with a distinguished one-parameter group of inner Lie automorphisms, we can speak of a "Hamiltonian" and "Hamiltonian equation of motion". We have good reasons for taking the Lie algebra structure for granted. The serious problem is the choice of additional structures, the Lie algebra alone being far from sufficient (and the quotation marks above more than justified).

The Lie algebra in itself is alien to the concept of states of a physical system. The states are usually thought of as positive linear functionals over the space of the variables with the implication that we know how the positivity is to be defined. Sure enough we know: the space of variables itself must be equipped with a positive cone. The point is that the Lie algebra structure does not generate non-trivial cones for an obvious reason – the Lie product is antisymmetric and therefore all Lie squares are equal to zero. The standard ordering of the associative algebras through the cone of the algebraic squares is not applicable to the Lie algebra context. Nonetheless, there are privileged candidates for cones turning (at least some of) the Lie algebras into ordered Lie algebras: the invariant cones which by definition transform onto themselves under the action of the group of the Lie algebra automorphisms. What makes them interesting is a sort of hereditary property – any algebraically generated cone is necessarily invariant. The Lie algebras give us a major example demonstrating that the converse is not true – there may be invariant cones not directly linked to the algebraic structure. There exists a complete classification of the invariant cones in finite dimensional Lie algebras; this problem was solved in the 1980s, but the study of the infinite dimensional case has not yet been initiated. The invariance property is a strong requirement even if it is restricted (as is usually done) to the group of the inner Lie automorphisms; it singles out a particular class of Lie algebras and prescribes a very specific texture to the invariant cones in them.

Finally, the positive cones in the conventional models of Hamiltonian mechanical systems are invariant with respect to the corresponding Lie algebras. This is equally true for the cone of the none-negative functions on the classical phase space (with respect to the Poisson brackets), and the cone of the positive self-adjoint operators (with respect to the operator commutator). Thus, the Lie algebras with invariant cones emerge as a sound basis for

an abstract model of general Hamiltonian systems. The invariance property relating the two (otherwise independent) structures allows immediate physical interpretation. It simply means that the group of the Lie automorphisms is contained in the group of the order automorphisms which, in turn, implies that the group of the common automorphisms is in a sense maximal. The group of the common automorphisms is actually the group of the symmetries of the physical system and any model of a general dynamical system must possess a sufficiently large symmetry group. Neither "sufficiently large" nor "maximal" is a well-defined term and the invariance of the cone is a property open to further discussions.

One last ingredient is needed, a variable identically (in all states) equal to 1, and it will be represented by an order unit. We are now ready for our first definitions.

General model (GM). The space of the variables of a Hamiltonian dynamical system is a real Lie algebra with a positive cone and order unit, both invariant with respect to the group of the Lie automorphisms. The states are normed positive linear functionals over the space of the variables. The values of the states at the elements of the Lie algebra are identified with the mean values of the corresponding variables. The time evolution is represented by a one-parameter group of inner Lie automorphisms.

In what follows some notations will be helpful.

Space of variables: $(A, A^+, e, [., .], \omega_t, ...)$ where A is a real linear space, A^+ is a positive cone, e is an order unit, $[., .]$ is a Lie product, and ω_t is the one-parameter dynamical group. The three dots indicate everything that must be added when we want to specify a certain class of systems or a concrete system.

Invariance requirement: $Aut(A, [., .]) \subseteq Aut(A, A^+, e)$.

Space of states: (V, V^+, K) where V is the dual space of A, V^+ is the dual cone of A^+, and the set of states $K \subset V^+$ satisfies $K(e) = 1$ (hence K is a base for V^+).

Dynamics: The condition that ω_t is in $Inn(A, [., .])$ implies the existence of a Hamiltonian and a Hamiltonian equation of motion.

The definitions in (GM) are extracted from the usual Hamiltonian type dynamical models but we have discarded everything that reflects the specific character of the classical or quantum systems. Our ultimate program assumes that the further specification of (GM) – up to describing concrete physical systems – should be carried out without leaving the language of the invariantly ordered Lie algebras.

III. HIERARCHY OF HAMILTONIAN SYSTEMS

Very little can be done in the framework of (GM) without an advanced theory of invariant cones in infinite dimensional Lie algebras. However, some developments in the theory of ordered linear spaces can be immediately incorporated into (GM) in order to bring the model closer to the known algebraic descriptions.

The presence of order unit e in A^+ and base K in V^+ suggests the first step towards stronger formulations.

Property (P1). The space (A, A^+, e) is an order-unit space (and therefore (V, V^+, K) is a base-norm space).

The terms "order-unit space" and "base-norm space" imply a little bit more than just existence of order unit or base, respectively. The order-unit space and the base-norm space are Banach spaces with norms determined through the order. The transition from (GM) to (GM)+(P1) essentially means that we choose to deal with bounded variables. A dual pair of an order-unit space and a base-norm space has been widely discussed in the early 1970s as a general framework for statistical physical systems.

The boom of the theory of ordered linear spaces in the 1970s culminated in a series of results with a key role for our purposes. It was shown that a certain class of order-unit spaces carry a complete spectral theory, a far-reaching generalization of the operator or algebraic spectral theory. The additional properties sufficient for the existence of spectral theory are usually introduced as "spectral duality".

Property (P2). The spaces (A, A^+, e) and (V, V^+, K) are an order-unit space and a base-norm space in spectral duality.

With (GM)+(P2) we reach a level of completeness which roughly corresponds to the Von Neumann algebra models, avoiding any reference to associative algebraic structures. The elimination of the associative algebras is not fictitious as one might think if one remembers that every Lie algebra can be embedded into an associative algebra. The enveloping associative algebra is irrelevant in our context since it fails to respect the presence of the second structure – the invariant order relation. At the level of (GM)+(P2) the variables are genuine random variables on Boolean event spaces, the events themselves related to the corresponding spectral resolutions.

The model (GM)+(P2) splits further into two categories of Hamiltonian systems that can be given the names "classical" and "quantum", respectively. The defining property of the classical systems is a lattice order in (A, A^+, e) while the antilattice order can be used as a characteristic of the quantum systems. The formulation (GM)+(P2) itself is actually the first unified description of classical and quantum systems which is reasonably complete and satisfies some natural aesthetic requirements.

Notice, however, that this method of specifying the classical and quantum systems is ineffective outside (GM)+(P2); it does not work in (GM)+(P1), let alone in (GM).

IV.   ORDERED LINEAR SPACES WITH INVARIANT LIE PRODUCTS, OR BEHIND THE LOOKING-GLASS

At the very beginning of Section II there is an element of arbitrariness that virtually predetermines (GM). There are no compelling reasons to start with a Lie algebra and ask which is the best way of introducing order relations into it. There is another option that must not be ignored – to start with an ordered linear space and look for Lie products consistent with the order relation. The order itself does not imply Lie algebra structure and we have no choice but to resort again to invariance requirements in terms of the two automorphism groups. This time, however, the inclusion relation between them is reversed – the group of the order automorphisms should be contained in the group of the Lie automorphisms. Instead of Lie algebras with invariant cones, we introduce ordered linear spaces with invariant Lie products.

Treating order and Lie product on an equal footing, we can simply say that there are two extreme cases when the group of their common automorphisms is maximal: either $Aut(A, [.,.]) \subseteq Aut(A, A^+, e)$ (invariance of the cone or, briefly, C-invariance), or $Aut(A, [.,.]) \supset Aut(A, A^+, e)$ (invariance of the Lie product or, briefly, L-invariance). The C-invariance is representative of the ordinary Hamiltonian systems as they are described by (GM) (as it is, the C-invariance should be slightly relaxed replacing $Aut(A, [.,.])$ by $Inn(A, [.,.])$). The L-invariance is something new and a series of questions arise. Do L-invariant systems exist? If so, what can we say about their behaviour? Are they classical or quantum systems or something else? At present no definite answers are possible. We argue that the most plausible answers are: L-invariant Hamiltonian systems exist, they reveal unmistakably quantum behaviour and – this is the crucial point – they are likely to offer an alternative description for what we now regard as quantum physical systems.

The distinguishing characteristic of the hypothetical L-invariant counterpart of (GM) is the appearance of a family of cones with a common order unit; we have to deal with many-ordered Lie algebra $(A, \varphi A^+, e, [.,.], \omega_t, ...)$ where A^+ is the original cone and φ is in the

larger group $Aut(A, [., .])$. Suppose now that A^+ is a lattice cone and $(A, \varphi A^+, e)$ is a family of vector lattices with a common order unit. The many-ordered Lie algebra transforms then into Lie algebra with many-valued commutative associative multiplication (the lattice order induces a unique associative algebraic structure which is necessarily commutative, the cone of the algebraic squares coinciding with the original lattice cone, and the algebraic unit coinciding with the order unit). In more general (non-lattice) cases no associative algebraic structure is implied but the many-orderedness still persists.

It is the many-orderedness of the space of the variables that suggests quantum behaviour of the L-invariant systems. In particular, the uncertainty relations – the trade-mark of the quantum systems – are inevitable whenever we speak of states over a many-ordered space of variables. Let us turn again to Lie algebras ordered by a family of lattice cones with common order unit, the L-invariant counterpart of the space of classical variables. Each cone of the family possesses its own dual cone with a simplex $\varphi' K$ as a base. The only contender for the set of states is the intersection of the sets $\varphi' K$. Thereby the extreme points of K and all its images $\varphi' K$ (the classical pure states) lose their status of states because they fail to satisfy the stronger positivity requirements. This process destroys the simplicial geometry of the original set K and invariably results in uncertainty relations for some pairs of variables.

Thus, when we reformulate the hierarchy of the Hamiltonian models from Section III to make them fit the L-invariance what we do is increasing the number of the variables (they depend now on additional parameters describing the family of cones) and reducing the set of states. Both changes indicate essentially quantum nature of the L-invariant systems. The temptation is strong to introduce the L-invariant systems – and in particular the vector lattices with invariant Lie products – as the quantum counterpart of the classical systems (Lie algebras with invariant lattice cones according to the classical version of (GM)+(P2)). The classical and quantum systems would then emerge as mirror images of one another, the transition between them being reduced to reversing the inclusion relation between the two automorphism groups.

To this end, however, we have to remove a serious obstacle: the L-invariant definitions are in a direct conflict with the present-day quantum theory which is based on a single invariant non-lattice (in fact antilattice) cone in the space of the variables (properties adopted by the quantum version of (GM)+(P2)). This is a major challenge to the viability of our project. What we are able to do now is singling out a class of L-invariant systems that could be made consistent with the conventional quantum models. Our starting point is the awareness that the L-invariant lattice model and the conventional (C-invariant,non-lattice) quantum theory cannot be considered equally complete if they are meant to refer to the same kind of physical objects. One of them must be a coarsened, factorized picture of the other. Which one?

Let us look at the problem from a formal standpoint. Constructing L-invariant models, we combine lattice order with invariant Lie product but we need not stop there, we may go on and ask whether the resulting Lie algebra, in turn, admits invariant cones. If it does, beside the original L-invariant model naturally appears a C-invariant model (possibly more than one). The transition to the derivative (presumably non-lattice) C-invariant model is a sort of factorization effacing some properties of the original L-invariant structure. In particular, the many-valued commutative multiplication in the space of the variables is irretrievably lost.

Thus, the C-invariant language can be regarded as providing a simplified description of

the hypothetical richer L-invariant quantum models. We will take the risk to attach the label "quantum" to the factorizable L-invariant lattice system and regard the C-invariant quantum version of (GM)+(P2) (and hence all operator or other conventional algebraic models) as approximations ignoring the many-valued commutative multiplication. The conventional non-commutativity (when it exists at all) is nothing but a distorted reminiscence of the lost many-valued commutativity. Much stronger but still plausible hypothesis is that the factorizations of the L-invariant quantum systems are actually LC-invariant, the two automorphism groups essentially coinciding. There is some evidence, indeed, that the typically quantum (factor-like) associative algebraic models exhibit such two-sided invariance.

Clearly, this classification scheme can be extended to the whole hierarchy of Hamiltonian systems: the C-invariant systems are called classical, the (factorizable) L-invariant systems are called quantum, and all the differences between then can be traced to the different direction of the inclusion relation between the two relevant automorphism groups. To avoid psychological difficulties, the LC-invariant systems may be separated into a special category encompassing the factorizations of the newly introduced quantum systems (with many-orderedness deleted). From practical point of view, the striking conclusion is that the standard operator quantum formalism and its algebraic generalizations take an intermediate position – no longer classical but not fully quantum.

Like Lewis Carroll's little Alice, we yielded to the temptation to reverse the invariance requirement and go through the Looking-Glass into the dreamland of the L-invariant systems. It is uncharted virgin territory; the L-invariant systems are legitimate mathematical objects – this is practically everything we know about them with certainty.

V. CONCLUDING REMARKS

The framework delineated by (GM) and its L-invariant counterparts seems to exhaust the potential of the Hamiltonian formalism for defining dynamical systems in terms of variables and states. The mathematical prerequisites for implementing such a program are far from complete. The invariant cones in infinite dimensional Lie algebras and the ordered linear spaces with invariant Lie products will remain on the agenda for the next few decades. The program itself adds weight to the suspicion that the non-commutative associative algebraic structure is a fallacious beacon in the search for quantum extensions of the classical theory. At the best it appears to be unnecessarily restrictive, and at the worst it stops halfway between the classical and quantum systems with no chance to move a step further without radical remodeling. Is a major part of the road to a satisfactory quantum theory still lying ahead?

VI. COMMENT ON THE SOURCES

The study of invariant cones in Lie algebras is initiated by Vinberg [1] and Paneitz [2]. Hilgert and Hofmann achieve exhaustive solutions for arbitrary finite dimensional Lie algebras [3], [4]. A pair of order-unit space and base-norm space as a suitable tool describing statistical physical systems is discussed by Davies and Lewis [5] and Edwards [6]. The main contribution to the spectral theory in the context of ordered linear spaces is made by

Alfsen and Shultz [7]. Let us remark that terms like "non-commutative spectral theory" are misleading – the spectral duality does not presuppose associative algebraic structures (commutative or not). Other versions are proposed by Abbati and Manià [8] and Riedel [9]. They all are generalizations of the spectral theory for vector lattices developed by Freudenthal [10]. The antilattice geometry of the cone of the positive self-adjoint operators is established by Kadison [11]. The combination of the geometric spectral theory of Alfsen and Shultz with the theory of invariant cones in Lie algebras was first propounded by the author in [12]. The L-invariant extensions appeared in [13], [14].

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